Lecture 22 Summary

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1 The Josephson Junction in a Magnetic Field

At this point we have the dc Josephson effect, which is a spontaneous Cooper pair curent that flows between two superconductors separated by a weak link as $J_s = J_c \sin(\theta_1 - \theta_2)$, where J_s is the super-current density, J_c is the critical current density (dependent on the barrier height and thickness), and $\theta_1 - \theta_2$ is the difference in phases of the macroscopic quantum wave functions in the two superconductors.
Now we wish to include the effect of a magnetic field on the Josephson junction. We shall assume that the superconducting banks remain in the Meissner state and look at the effects of the field on the junction properties. To do this, we appeal to the gauge invariance of the observables, namely $|\Psi(r,t)|^2$ and $J_s = \frac{q^* n^*}{m^*}(\hbar \overrightarrow{\nabla} \theta - q^* \overrightarrow{A})$, and demand that their values not depend on a choice of gauge for $\overrightarrow{A}$ and $\overrightarrow{B}$. A new gauge can be created as $\overrightarrow{A}' = \overrightarrow{A} + \overrightarrow{\nabla}\chi(r)$, where $\chi(r)$ is an arbitrary scalar function of position. This will leave J_s and $|\Psi(r,t)|^2$ invariant if we also modify the phase of the macroscopic quantum wavefunction as $\theta' = \theta + \frac{q^*}{\hbar}\chi(r)$. Using $q^* = -2e$, we have a new phase difference on the junction $\gamma = \theta_1' - \theta_2' - \frac{2\pi}{\Phi_0}(\chi_1 - \chi_2)$. Writing the difference in χ as the line integral of $\overrightarrow{\nabla}\chi(r)$, we get this expression for the gauge-invariant phase difference γ as, $\gamma = \theta_1 - \theta_2 - \frac{2\pi}{\Phi_0}\int_1^2 \overrightarrow{A} \cdot \overrightarrow{dl}$. One can show that the change in gauge introduced above leaves this quantity unchanged.
Now we have the result that $J_s = J_c \sin(\gamma)$, with $\gamma = \theta_1 - \theta_2 - \frac{2\pi}{\Phi_0}\int_1^2 \overrightarrow{A} \cdot \overrightarrow{dl}$ as a more complete expression for the dc Josephson effect. We can see that an applied magnetic field has the ability to modify the supercurrent flowing through the junction.

2 The ac Josephson Effect

We wish to understand the dynamics of a Josephson junction. If a supercurrent does not cause the phase difference γ to "wind", then what does?
Take the time derivative of the gauge invariant phase difference,
$\frac{\partial \gamma}{\partial t} = \frac{\partial \theta_1}{\partial t} - \frac{\partial \theta_2}{\partial t} - \frac{2\pi}{\Phi_0}\frac{\partial}{\partial t}\int_1^2 \overrightarrow{A} \cdot \overrightarrow{dl}$

Back in HW2 you derived an expression for the dynamics of the phase of the macroscopic quantum wavefunction $\Psi(r,t) = \sqrt{n^*}e^{i\theta(r,t)}$, where n^* is assumed independent of space and time, as,
$-\hbar\frac{\partial\theta}{\partial t} = \frac{1}{2n^*}\Lambda J_s^2 + q^*\phi$, where ϕ is the electrostatic potential.
Using this in the expression for $\frac{\partial\gamma}{\partial t}$, assuming that the current is continuous across the junction, and that the difference in scalar potential can be written as the line integral of the gradient, we arrive at,
$\frac{\partial\gamma}{\partial t} = \frac{2\pi}{\Phi_0}\int_1^2 \left(-\overrightarrow{\nabla}\phi - \frac{\partial\overrightarrow{A}}{\partial t}\right)\cdot\overrightarrow{dl}$.
The quantity in parentheses is the total electric field, that due to both scalar and vector sources. Hence we have
$\frac{\partial\gamma}{\partial t} = \frac{2\pi}{\Phi_0}\int_1^2 \overrightarrow{E}\cdot\overrightarrow{dl}$.
This integral is just the potential difference between the superconductors, yielding the famous ac Josephson effect expression:
$\frac{\partial\gamma}{\partial t} = \frac{2\pi}{\Phi_0}\Delta V$. Hence, by applying a dc potential difference across the junction you can get the gauge-invariant phase difference to "wind".

3 Circuit Model of a Josephson Junction

One can look at a Josephson junction as a circuit element. By integrating the current density over the entire junction one can relate the total current through the device to the gauge-invariant phase difference (GIPD) across the device: $I = I_c\sin(\gamma)$. In the case of a voltage drop V across the junction, the GIPD will wind as $V = \frac{\Phi_0}{2\pi}\frac{d\gamma}{dt}$.

Suppose a static dc voltage V_{dc} is applied to the junction. The GIPD can be found from integration: $\gamma(t) = \gamma(0) + \frac{2\pi}{\Phi_0}V_{dc}t$. This leads to an alternating current through the junction, given by $I = I_c\sin(2\pi f_J t + \gamma(0))$. The Josephson frequency is $f_J = \frac{V_{dc}}{h/2e} = 483.6$ (THz/V) $V_{dc} = 483.6$ (MHz/μV) V_{dc}. The JJ acts as a very precise voltage-to-frequency transducer and vice versa.
The NIST (and world) voltage standard is based on generating a precise mm-wave signal (at about 90 GHz) and shining it on a series array of Josephson junctions that are designed to yield a total dc voltage drop of precisely 1 volt.
Going the other way, one can use intrinsic Josephson junctions that occur in layered high-Tc cuprates (like Bi-Sr-Ca-Cu-O), biased by a dc voltage, to create a coherent mm-wave and THz source. The output frequency can be tuned by about 10 to 20% by altering the dc voltage. In principle the output power should scale with the number of junction layers squared, and it does. However as the stacks of JJs grow thicker they fail to operate properly due to internal heating and other source of nonlinearity.
These applications are illustrated on the class web site.

4 Josephson Junction Critical Current Modulation with a Magnetic Field

Consider an SIS Josephson junction with barrier thickness $2a$ and superconducting electrodes with thickness much larger than the effective penetration depth. The middle of the barrier lies in the yz plane. Apply a dc magnetic field through the barrier parallel to the electrodes $\overrightarrow{B} = B\widehat{y}$.
By drawing a contour of width dz in the z-direction and height much greater than the effective penetration depths in the two electrodes, one can follow the evolution of the phase of the macroscopic quantum wavefunction on the contour and deduce a differential equation for the evolution of the gauge-invariant phase difference along the junction in the z-direction:
$\frac{\partial\gamma}{\partial z} = \frac{2\pi d_{eff}}{\Phi_0} B$
Here $d_{eff} = 2a + \lambda_1 + \lambda_2$ is the "magnetic thickness" of the barrier. This was derived under the assumption that the junction is "short" (width $L < \lambda_J$, where λ_J is derived below), and makes no significant alteration to the applied magnetic field through screening.
Integrating this result along the width of the junction yields,
$\gamma(z) = \gamma(0) + 2\pi \frac{d_{eff} L B}{\Phi_0} \frac{z}{L}$. This expression contains the total flux through the junction $\Phi_J = B d_{eff} L$ divided by the flux quantum,
$\gamma(z) = \gamma(0) + 2\pi \frac{\Phi_J}{\Phi_0} \frac{z}{L}$.

Integrating the current density over the area of the junction yields the total current through the junction,
$I = I_c \sin(\gamma(0)) \frac{\sin(\pi\Phi_J/\Phi_0)}{\pi\Phi_J/\Phi_0}$, where $I_c \equiv J_c W L$. This is the famous magnetic diffraction curve for modulation of the critical current of a JJ with external in-plane magnetic field.
At $\Phi_J = 0$ the current $J = J_c \sin(\gamma(z))$ is uniform over the junction and it has a maximum critical current. At $\Phi_J/\Phi_0 = 1$ there is a linear increase of γ from $\gamma(0)$ to $\gamma(0) + 2\pi$ from one edge of the junction to the other, creating a single-period sinusoidal oscillation of current through the junction. This results in zero net current through the junction.
The resemblance of the critical current diffraction pattern to single-slit diffraction in wave optics shows the analogy to interference created by spatial variation of a phase, $\gamma(z)$ in this case.

The "long junction" case considers the effect of self-generated modifications of the applied magnetic field due to screening. Starting with the differential equation derived above, $\frac{\partial\gamma}{\partial z} = \frac{2\pi d_{eff}}{\Phi_0} B$, now assume that $B = B_{ext} + B_{self}$ and that the total magnetic field must satisfy Ampere's law: $\overrightarrow{\nabla} \times \overrightarrow{B} = \mu_0 \overrightarrow{J} + \epsilon\mu_0 \frac{\partial \overrightarrow{E}}{\partial t}$. Assuming a static situation and that the edges of the junction are far away so that $\overrightarrow{B} = B_y(z)\widehat{y}$ only, then Ampere's law becomes $\frac{\partial B_y(z)}{\partial z} = -\mu_0 J_x(z)$. Combining this with the differential equation for $\gamma(z)$ yields the 1D sine-Gordon equation,

$\frac{d^2\gamma}{dz^2} = \frac{1}{\lambda_J^2}\sin(\gamma(z))$,
where $\frac{1}{\lambda_J^2} = \frac{2\pi\mu_0 d_{eff} J_c}{\Phi_0}$, which defines the Josephson penetration depth $\lambda_J = \sqrt{\frac{\Phi_0}{2\pi\mu_0 d_{eff} J_c}}$. Note that the Josephson penetration depth is typically much larger than the London penetration depth at the same T/T_c. This makes Josephson vortices very elongated in the junction direction, as shown on the class web site.